

A Problem List (with Answers)

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Some of these problems will look less familiar than the text's end-of-section problems, mainly because they are not at the end of a section. But they involve no new notions, and it is worthwhile to solve them all. Don't worry unduly if a few problems seem difficult—some are meant to be.

Answers (not solutions) to most of the problems are at the end of the list.

1. Let $f(x) = \frac{1}{x^3 - x^2}$. Use the *definition* of the derivative to find $f'(5)$. (We can easily find the answer by using the quotient rule, but that is not what is asked for.)
2. If $h(x)$ is a function, then $h'(x)/h(x)$ is called the *relative rate of change* of $h(x)$.
- a) Show that the relative rate of change of $f(x)g(x)$ is equal to the relative rate of change of $f(x)$ plus the relative rate of change of $g(x)$.
- b) Show that the relative rate of change of $f(x)/g(x)$ is equal to the relative rate of change of $f(x)$ minus the relative rate of change of $g(x)$.
3. Let A be the point $(1, 1)$, and let ℓ be the normal to the curve $y = x^2$ at the point A . The line ℓ meets $y = x^2$ at A and at a second point B . Find the slope of the tangent line to $y = x^2$ at B . (The *normal* to a curve $y = f(x)$ at the point P on the curve is the line through P which is perpendicular to the tangent line to $y = f(x)$ at P .)
4. Let c be a positive constant. Show that the sum of the x -intercept and the y -intercept of any tangent line to the curve $x^{1/2} + y^{1/2} = c^{1/2}$ is equal to c .
5. (a) Find the equation of the tangent line to the curve $3^x - 2^y = 1$ at the point $(2, 3)$.
- (b) By expressing $\frac{dy}{dx}$ as a function of x , or otherwise, find $\lim_{x \rightarrow \infty} \frac{dy}{dx}$.
6. Find all values of a for which the tangent line to $y = \arctan x$ at $x = a$ is parallel to the tangent line to $y = \ln(x + 2)$ at $x = a$.
7. A particle is travelling along the curve $y = x^2 - 5$. At the instant that the particle reaches the point $(3, 4)$, the particle's distance from the origin is increasing at the rate of 2 units per second. How fast is the particle's x -coordinate increasing at this instant?
8. Find all values of a such that the polynomial $x^3 - 3x^2 + ax + 1$ has a local maximum.

9. Find the coordinates of the highest point on the curve $x^2 + xy + y^2 = 1$.
10. The following facts are known about the function $f(x)$: (i) $f(2) = 4$ and (ii) $f'(x) = (x^4 + 1)^{-1}$ for all x .
- (a) Use a linear approximation to estimate $f(2.1)$
- (b) Is your estimate larger or smaller than the true value of $f(2.1)$? Explain.
11. You have invested \$3000 in an account at the Bank of Calculus. The nominal yearly interest rate is 4.5%, and compounding is (of course) done continuously. How fast is your bank account growing at the instant that the amount in the account reaches \$6000?
12. A company estimates that if it spends x thousand dollars in advertising, where $0 \leq x \leq 16$, then it will sell a quantity $Q(x)$ of its product, where $Q(x) = 2000 - 2x^3 + 60x^2 + 750x$. At what value of x does the rate of change of $Q(x)$ (with respect to x) reach a maximum? (This value of x is sometimes called the *point of diminishing returns*.)
13. I borrowed \$6000 from a daughter who had taken Math 104, and who therefore had a preference for continuous compounding. Two years later, she informed me that my debt had grown to \$7260. So I paid her \$4260, and paid off the rest of my debt one year later. What was that final payment?
14. The cost $C(q)$ of producing q kilograms of a certain pharmaceutical is given by $C(q) = 0.0001q^4 - 0.24q^2 + 500q + 9000$. At what level of production does the marginal cost start to increase?
15. Tiffany sleeps 8 hours per day. She has a summer job that allows her to work any number of hours per day (up to 8 hours) at \$6 per hour. The remaining hours are spent doing mathematics. Her mother gives her an allowance of \$6 per day.

The amount of daily happiness that Tiffany gets from getting y dollars and spending h hours doing mathematics is given by $U(y, h)$, where

$$U(y, h) = y^{1/3}h^{2/3}.$$

How many hours per day should she work to maximize her happiness?

16. A well-dressed mathematician owns \$3,000 dollars' worth of clothes. These clothes are depreciating: t years from now, their value will be $3000e^{-0.16t}$. The mathematician also has \$5,000 invested in a GIC that yields a nominal interest rate of 2%, with continuous compounding. The mathematician has no other assets, and all of the mathematician's paltry income is spent on food, shelter, and tobacco. When will the mathematician's *total* assets reach a minimum? Show that your answer really does give the minimum.

17. A high speed train is traveling at 2 km/min along a straight track. The train is moving away from a movie camera which is located 0.6 km from the track.

The camera keeps turning so as to always point at the front of the train. How fast (in radians per minute) is the camera rotating when the front of the train is 1 km from the camera?

18. After half a lifetime at CBU, a professor's life savings consist of two registered savings accounts. The first account now has 400 dollars, and earns interest at a nominal 2%, continuously compounded. The second account now has 350 dollars, and earns interest at a nominal 6%, continuously compounded.

a) How much time will it take for the two accounts to have exactly the same amount of money in them? A calculator-ready *expression* for the answer is enough.

b) At what rate is the professor's wealth increasing right now?

19. Suppose that the function y satisfies the differential equation $\frac{dy}{dt} = -5y$, and let $z = y^2$. Then z satisfies a differential equation of the form $\frac{dz}{dt} = f(z)$. Find a formula for $f(z)$.

20. Find all values of the constants A and k for which the function given by $y = Ae^{kt}$ satisfies the differential equation $\frac{d^2y}{dt^2} = \frac{dy}{dt} + y$.

21. We want to travel to the right from the origin O along the x -axis, then straight up until we meet the curve $y = 4 - \frac{x^2}{9}$, then straight left until we meet the y -axis, and finally straight down until we get back to O . Find the total length of the longest possible such trip.

22. Use calculus to find which is bigger, $(\sqrt{1000001} + \sqrt{999999})/2$ or 1000.

23. A rectangle is inscribed in a circle of radius 1, and an identical rectangle is also inscribed in the circle, in a direction perpendicular to the first rectangle. Together, the two rectangles form a cross, and the part they have

in common is a square. What should be the dimensions of the rectangles so that the area which is inside the cross but outside the central square is as large as possible?

24. Let a be a positive constant. Find the maximum value of $(a/x)^x$ as x ranges over the positive reals.
25. A particle is travelling in the plane. At any time t , its x -coordinate is given by $x = \sin^2 t$, and its y -coordinate is given by $y = \cos t$. What is the closest it ever gets to the point $(0,0)$?
26. In a simple camera lens of focal length f , if an object is at distance p from the plane of the lens, then its image is at distance q from the plane of the lens, where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}.$$

A certain lens has focal length f equal to 10 centimeters. An object is moving away from the lens at 2.4 centimeters per second. How fast is its image moving away from the lens at the instant that the object is 30 centimeters from the lens?

27. Points P and Q are chosen on the curve $x^4 + 16y^4 = 1$ in such a way that the distance PQ is as large as possible. Find that distance.
28. Alicia wanted to approximate a solution of the equation $f(x) = 0$, and made the initial estimate $x_0 = 2$. The number $f(2)$ turned out to be positive. The Newton Method gave x_1 as next estimate, where $x_1 = 2.05$. Let $g(x) = \ln(f(x))$. Find $g'(2)$.

29. Suppose that $f(3) = 9$ and $f'(3) = 4$. Show that

$$\lim_{x \rightarrow 3} \frac{\sqrt{f(x)} - 3}{x - 3} = \frac{2}{3}.$$

(We are not allowed to assume that $f'(x) \rightarrow 4$ as $x \rightarrow 3$, since there are functions f such that $f'(3) = 4$ but $f'(x) \not\rightarrow 4$ as $x \rightarrow 3$.)

30. At what first-quadrant point on the parabola $y = 1 - x^2$ does the tangent line have the property that it cuts from the first quadrant a triangle of minimum area?
31. According to the Special Theory of Relativity, a body that has rest mass m_0 and is moving at velocity v has relativistic mass m_v given by

$$m_v = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where c is the velocity of light.

A sprinter with rest mass 70 kilograms is moving at velocity $v = 0.01$ kilometers per second. Use a suitable linear approximation to the function $(1 - x)^{-1/2}$ to estimate $m_v - 70$, the relativistic change in mass of the sprinter. (Take c to be 300,000 kilometers per second.)

32. Three of the sides of a trapezoid have length 1. What should be the length of the fourth side if the area of the trapezoid is to be as large as possible?

33. A water tank has the shape of a circular cone with top radius 6 metres and vertical depth 9 metres. Water is being poured into the tank at a constant rate of 14π cubic metres per hour, and is simultaneously leaking out of a large hole at the bottom at a rate of ky cubic metres per hour when the depth of water in the tank is y metres. Here k is a constant. When the water is 3 metres deep in the tank, its surface is rising at a rate of 2 metres per hour. Find k . What is the maximum depth the water in the tank can reach?
34. Let $f(x) = (x+1)(x+2)(x+4)(x+8)(x+16)$. Find $f'(0)$. Please simplify. *Hint:* Let $g(x) = \ln(f(x))$.

35. Let a , p , and q be positive constants. Find the largest value taken on by $(a+x)^p(a-x)^q$ as x ranges over the interval from $-a$ to a .
36. When a fish swims upstream for t hours at speed s , *relative to the water*, the total amount of energy it expends is approximately equal to ks^3t for some constant k .
- Assume that salmon swim in a way that minimizes the total energy expended. How long will it take a salmon to go 240 kilometers upstream against a steady 6 kilometer per hour current?
37. (a) In the picture below, $PX = x$, $PA = a$, and $PB = b$. Express $\angle AXB$ in terms of a , b , x , and the arctan function.
- (b) Find the value of x that maximizes $\angle AXB$.

Answers

3. -3 (Correction: WK)
5. (a) $y = mx + (3 - 2m)$ where $m = (9 \ln 3)/(8 \ln 2)$; (b) $(\ln 3)/(\ln 2)$
6. $a = (1 \pm \sqrt{5})/2$
7. $10/27$ units per second
8. all a such that $a < 3$
9. $(-1/\sqrt{3}, 2/\sqrt{3})$
10. (a) $4 + (0.1)/17$; (b) larger because curve is concave down
11. \$270 per year (Correction: WK)
12. $x = 10$
13. \$3300
14. 20 kg
15. $14/3$ hours
16. $(\ln(4.8))/(0.18)$
17. 1.2 radians per minute
18. (a) $(\ln(400/350))/(0.04)$; (b) \$29 per year (Correction: WK)
19. $f(z) = -10z$
20. $A = 0$, k anything; or $k = (1 \pm \sqrt{5})/2$, A anything

21. $25/2$
22. 1000
23. $\sqrt{2 + \sqrt{2}}$ by $\sqrt{2 - \sqrt{2}}$
24. $e^{a/e}$
25. $\sqrt{3}/2$
26. 0.6 cm per second
27. $17^{1/4}$
28. -20 (Correction: FC)
30. $(1/\sqrt{3}, 2/3)$
31. roughly 3.9×10^{-14} kg
32. 2
33. $2\pi; 7$
34. 1984
35. $p^p q^q (2a)^{p+q} / (p+q)^{p+q}$
36. 80 hours
37. (a) $\arctan(b/x) - \arctan(a/x)$; (b) \sqrt{ab}